

EXPLICIT COMPUTATION OF THE CHERN CHARACTER FORMS

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ABSTRACT. We propose a method for explicit computation of the Chern character form of a holomorphic Hermitian vector bundle (E, h) over a complex manifold X in a local holomorphic frame. First, we use the descent equations arising in the double complex of (p, q) -forms on X and find explicit degree decomposition of the Chern-Simons form cs_k associated to the Chern character form ch_k of (E, h) . Second, we introduce the ‘ascent’ equations that start from the $(2k - 1, 0)$ component of cs_k , and use Cholesky decomposition of the Hermitian metric h to represent the Chern-Simons form, modulo d -exact forms, as a ∂ -exact form. This yields a formula for the Bott-Chern form bc_k of type $(k - 1, k - 1)$ such that $ch_k = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial bc_k$. Explicit computation is presented for the cases $k = 2$ and 3 .

1. INTRODUCTION

Let V be a C^∞ -complex vector bundle with a connection $\nabla = d + A$ over a smooth manifold X . The Chern character form $ch(V, \nabla)$ for the pair (V, ∇) is defined by

$$ch(V, \nabla) = \text{tr} \left\{ \exp \left(\frac{\sqrt{-1}}{2\pi} \nabla^2 \right) \right\}.$$

Here ∇^2 is the curvature of the connection ∇ , $\text{End } V$ -valued 2-form on X , and tr is the trace in the endomorphism bundle $\text{End } V$. The Chern character form is closed, $dch(V, \nabla) = 0$, and its cohomology class in $H^*(X, \mathbb{C})$ does not depend on the choice of ∇ (see, e.g., [1]).

Let ∇^0 and ∇^1 be two connections on V . In [2], S.S. Chern and J. Simons introduced secondary characteristic forms — the Chern-Simons forms $cs(\nabla^1, \nabla^0)$. They are defined modulo exact forms, satisfy the equation

$$(1) \quad dcs(\nabla^1, \nabla^0) = ch(V, \nabla^1) - ch(V, \nabla^0),$$

and enjoy a functoriality property under the pullbacks with smooth maps. When the bundle V is flat, putting $\nabla^1 = d + A$ and $\nabla^0 = d$ and using linear homotopy $A(t) = tA$ in the Chern-Weil homotopy formula, one obtains an explicit formula for the Chern-Simons form $cs(A)$ in terms of A .

Let (E, h) be a holomorphic Hermitian vector bundle — a holomorphic vector bundle of rank r over a complex manifold X , $\dim_{\mathbb{C}} X = n$, with a Hermitian metric h . Chern-Weil theory associates to any polynomial Φ on

$\mathrm{GL}(r, \mathbb{C})$, invariant under conjugation, a differential form $\Phi(\Theta)$ on X . Special case of this construction is the Chern character form $\mathrm{ch}(E, h)$, defined by

$$\mathrm{ch}(E, h) = \mathrm{tr} \left\{ \exp \left(\frac{\sqrt{-1}}{2\pi} \Theta \right) \right\} = \sum_{k=0}^n \mathrm{ch}_k(E, h).$$

Here Θ is the curvature of the canonical connection $d + \theta$ in E associated with the Hermitian metric h . In the local holomorphic frame, $\theta = h^{-1} \partial h$ and $\Theta = \bar{\partial} \theta$ (see, e.g., [1]).

Let h_1 and h_2 be two Hermitian metrics on a holomorphic vector bundle E over a complex manifold X . In the classic paper [3], Bott and Chern showed the existence of certain secondary characteristic forms, the Bott-Chern secondary forms $\mathrm{bc}(E, h_1, h_2)$. They are defined modulo ∂ and $\bar{\partial}$ -exact forms, satisfy the equation

$$\frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \mathrm{bc}(E, h_1, h_2) = \mathrm{ch}(E, h_1) - \mathrm{ch}(E, h_2)$$

and enjoy the functoriality property with respect to the pullbacks by holomorphic maps. Here the Chern character forms are computed for canonical connections in (E, h_1) and (E, h_2) . The Bott-Chern forms have been used in geometric stability [4, 5], in higher dimensional Arakelov geometry [6, 7] and in physics [8] (see also [9] for their application to differential K -theory).

However, it is difficult to obtain explicit formulas for the Bott-Chern forms. It is already mentioned in the remark in [3, Sect. 3] that even for a linear homotopy h_t of Hermitian metrics, the homotopy formula in Proposition 3.15 in [3] contains the inverse metrics through $\Theta_t = \bar{\partial}(h_t^{-1} \partial h_t)$ and does not allow to integrate over t in a closed form. As the result, it is difficult to get explicit formulas for the Bott-Chern forms in terms of Hermitian metrics h_1 and h_2 only (thus in [4] it is stated ‘‘One interesting feature is that we have an example of a variational problem with no simple explicit formula for the Lagrangian’’). This problem manifests itself even for the case when E is a trivial bundle with metrics $h_1 = h$ and $h_2 = I$, the identity matrix.

Here we show how using global coordinates on the space of Hermitian positive-definite matrices associated with the Cholesky decomposition, one can obtain explicit formulas for the Bott-Chern forms on trivial bundles. Namely, in Section 2 we find the explicit decomposition into (p, q) -forms of the Chern-Simons form cs_k associated to the Chern character form $\mathrm{ch}_k = \mathrm{ch}_k(E, h)$. It is done by solving the descent equations from the double complex of (p, q) -forms on X , applied to ch_k . In Section 3 we introduce the ‘ascent’ equations that start from the $(2k - 1, 0)$ component of cs_k , and use Cholesky decomposition of the Hermitian metric h to represent the Chern-Simons form, modulo d -exact forms, as a ∂ -exact form. This yields an explicit formula for the Bott-Chern form bc_k of type $(k - 1, k - 1)$ such that $\mathrm{ch}_k = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \mathrm{bc}_k$. It is obtained by repeatedly finding corresponding

∂ -antiderivatives and seems to be very non-local. For the case $k = 2, 3$ we show that in Cholesky coordinates these forms are given by explicit local formulas. We believe that this is the case for all k .

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2. DOUBLE DESCENT

2.1. Set-up. Let h be Hermitian metric in rank r trivial complex vector bundle over a complex manifold X (i.e., in general we consider a local holomorphic frame over some open neighborhood). Put (see, e.g., [1])

$$\theta = h^{-1}\partial h \quad \text{and} \quad \Theta = \bar{\partial}\theta.$$

We have the following useful formulas

$$(2) \quad \partial\theta = -\theta^2, \quad \bar{\partial}\theta = \Theta \quad \text{and} \quad \partial\Theta = [\Theta, \theta], \quad \bar{\partial}\Theta = 0,$$

where for matrix-valued differential forms we write AB instead of $A \wedge B$, etc. In particular, we have

$$(3) \quad \partial\Theta^k = [\Theta^k, \theta] \quad \text{and} \quad \partial(\theta\Theta^k) = -\theta\Theta^k\theta.$$

We have (using a ‘system of units’ such that $\frac{\sqrt{-1}}{2\pi} = 1$)

$$\text{ch}_k(h) = \frac{1}{k!}\omega_{k,k},$$

where

$$\omega_{k,k} = \text{tr } \Theta^k$$

is ∂ and $\bar{\partial}$ -closed real form of type (k, k) ; here and in what follows $\omega_{p,q}$ denotes a (p, q) -form. It follows from Poincaré lemma that locally there are forms $\omega_{k+l, k-l-1}$ such that

$$\begin{aligned} \omega_{k,k} &= \bar{\partial}\omega_{k,k-1}, \\ \partial\omega_{k,k-1} &= \bar{\partial}\omega_{k+1,k-2}, \\ &\vdots \\ \partial\omega_{2k-2,1} &= \bar{\partial}\omega_{2k-1,0}, \\ \partial\omega_{2k-1,0} &= 0. \end{aligned}$$

These descent equations¹ can be written succinctly as

$$(4) \quad \omega_{k,k} = (\bar{\partial} - t\partial)\left(\omega_{k,k-1} + t\omega_{k+1,k-2} + \cdots + t^{k-2}\omega_{2k-2,1} + t^{k-1}\omega_{2k-1,0}\right).$$

¹Compare with the double descent in [10] and with the holomorphic descent in [8].

Remark 1. Putting $t = -1$ we get

$$\omega_{k,k} = d\left(\omega_{k,k-1} - \omega_{k+1,k-2} + \cdots + (-1)^{k-2}\omega_{2k-2,1} + (-1)^{k-1}\omega_{2k-1,0}\right),$$

where $d = \partial + \bar{\partial}$. This gives an explicit decomposition of the Chern-Simons secondary form cs_k into (p, q) -degrees, $p + q = 2k - 1$:

$$(5) \quad \text{cs}_k = \frac{1}{k!} \left(\omega_{k,k-1} - \omega_{k+1,k-2} + \cdots + (-1)^{k-2}\omega_{2k-2,1} + (-1)^{k-1}\omega_{2k-1,0} \right).$$

It is easy to compute all these forms using (2)–(3) and equations

$$(6) \quad \partial\theta^2 = 0, \quad \bar{\partial}\theta^2 = [\Theta, \theta].$$

First, we observe

$$\omega_{k,k-1} = \text{tr}(\theta\Theta^{k-1})$$

and state the following result.

Lemma 1. *We have*

$$\partial\omega_{k,k-1} = \text{tr}(\theta^2\Theta^{k-1}) = \bar{\partial}\omega_{k+1,k-2},$$

where

$$\omega_{k+1,k-2} = \frac{1}{k+1} \text{tr} \left\{ \theta \left(\theta^2\Theta^{k-2} + \Theta\theta^2\Theta^{k-3} + \cdots + \Theta^{k-3}\theta^2\Theta + \Theta^{k-2}\theta^2 \right) \right\}.$$

Proof. Using (2)–(3), we get

$$\partial \text{tr}(\theta\Theta^{k-1}) = \text{tr}(-\theta^2\Theta^{k-1} - \theta(\Theta^{k-1}\theta - \theta\Theta^{k-1})) = -\text{tr}(\theta\Theta^{k-1}\theta) = \text{tr}\theta^2\Theta^{k-1}.$$

Next, using (2) and (6), we get

$$\begin{aligned} \bar{\partial} \sum_{i=0}^{k-2} (\theta\Theta^i\theta^2\Theta^{k-2-i}) &= \sum_{i=0}^{k-2} (\Theta^{i+1}\theta^2\Theta^{k-2-i}) - \sum_{i=0}^{k-2} (\theta\Theta^i(\theta\Theta - \Theta\theta)\Theta^{k-2-i}) \\ &= \sum_{i=0}^{k-2} (\Theta^{i+1}\theta^2\Theta^{k-2-i}) + \theta^2\Theta^{k-1} - \theta\Theta^{k-1}\theta, \end{aligned}$$

since the second sum telescopes. Using the cyclic property of the trace, we get the formula for $\omega_{k+1,k-2}$. \square

Observe that $\omega_{k,k-1}$ is a constant term a_0 in the polynomial

$$(7) \quad F_k(t) = \text{tr} \left\{ \theta(\Theta + t\theta^2)^{k-1} \right\} = a_0 + a_1t + \cdots + a_{k-1}t^{k-1},$$

while by Lemma 1,

$$\omega_{k+1,k-2} = \frac{1}{k+1}a_1.$$

This suggests to consider all coefficients a_l of $F_k(t)$ — differential forms of types $(k+l, k-l-1)$, $l = 0, 1, \dots, k-1$.

Lemma 2. *Put $G_k(t) = \text{tr}(\Theta + t\theta^2)^k$. We have*

$$\bar{\partial}F_k(t) - t\partial F_k(t) = G_k(t).$$

Proof. It follows from equations (2) and (6) that

$$\partial(\Theta + t\theta^2) = [(\Theta + t\theta^2), \theta] \quad \text{and} \quad \bar{\partial}(\Theta + t\theta^2) = t[(\Theta + t\theta^2), \theta],$$

which implies

$$\partial(\Theta + t\theta^2)^k = [(\Theta + t\theta^2)^k, \theta] \quad \text{and} \quad \bar{\partial}(\Theta + t\theta^2)^k = t[(\Theta + t\theta^2)^k, \theta].$$

Therefore,

$$\begin{aligned} \partial F_k(t) &= \text{tr} \left\{ -\theta^2(\Theta + t\theta^2)^{k-1} - \theta \left((\Theta + t\theta^2)^{k-1} \theta - \theta(\Theta + t\theta^2)^{k-1} \right) \right\} \\ &= \text{tr} \left\{ \theta^2(\Theta + t\theta^2)^{k-1} \right\} \end{aligned}$$

and

$$\begin{aligned} \bar{\partial} F_k(t) &= \text{tr} \left\{ \Theta(\Theta + t\theta^2)^{k-1} - t\theta \left((\Theta + t\theta^2)^{k-1} \theta - \theta(\Theta + t\theta^2)^{k-1} \right) \right\} \\ &= t \text{tr} \left\{ \theta^2(\Theta + t\theta^2)^{k-1} \right\} + \text{tr}(\Theta + t\theta^2)^k, \end{aligned}$$

so that $(\bar{\partial} - t\partial)F_k(t) = G_k(t)$. □

From here it is easy to find all descent forms $\omega_{k+l, k-l-1}$.

Proposition 1. *We have for all $k = 1, 2, \dots$,*

$$\omega_{k+l, k-l-1} = \frac{k!l!}{(k+l)!} a_l, \quad l = 0, 1, \dots, k-1.$$

In particular,

$$\omega_{2k-1, 0} = \frac{k!(k-1)!}{(2k-1)!} \text{tr} \theta^{2k-1}.$$

Proof. We observe that

$$\frac{dG_k}{dt}(t) = k \text{tr} \left\{ \theta^2(\Theta + t\theta^2)^{k-1} \right\} = k \partial F_k(t),$$

so that

$$G_k(t) = b_0 + k \partial a_0 \frac{t}{1} + k \partial a_1 \frac{t^2}{2} + \dots + k \partial a_{k-1} \frac{t^k}{k},$$

where $b_0 = \text{tr} \Theta^k = \omega_{k, k}$. Now it follows from Lemma 2 that

$$\bar{\partial} a_l = \left(\frac{k+l}{l} \right) \partial a_{l-1}, \quad l = 1, \dots, k-1,$$

and since $a_0 = \omega_{k, k-1}$, we easily obtain

$$a_l = \frac{(k+l) \dots (k+1)}{l!} \omega_{k+l, k-l-1}. \quad \square$$

Thus for $k = 1$ we have

$$\omega_{1,0} = \text{tr} \theta = \partial \log \det h \quad \text{and} \quad \omega_{0,0} = \log \det h,$$

whereas for $k = 2$

$$\omega_{2,1} = \text{tr}(\theta \Theta) \quad \text{and} \quad \omega_{3,0} = \frac{1}{3} \text{tr} \theta^3.$$

For $k = 3$ we have

$$\omega_{3,2} = \text{tr}(\boldsymbol{\theta}\boldsymbol{\theta}^2), \quad \omega_{4,1} = \frac{1}{2} \text{tr}(\boldsymbol{\theta}^3\boldsymbol{\theta}) \quad \text{and} \quad \omega_{5,0} = \frac{1}{10} \text{tr} \boldsymbol{\theta}^5,$$

and for $k = 4$ from Proposition 1 we obtain

$$\omega_{4,3} = \text{tr}(\boldsymbol{\theta}\boldsymbol{\theta}^3), \quad \omega_{5,2} = \frac{1}{5} \text{tr}(\boldsymbol{\theta}^3\boldsymbol{\theta}^2 + \boldsymbol{\theta}\boldsymbol{\theta}\boldsymbol{\theta}^2\boldsymbol{\theta} + \boldsymbol{\theta}\boldsymbol{\theta}^2\boldsymbol{\theta}^2), \quad \omega_{6,1} = \frac{1}{5} \text{tr}(\boldsymbol{\theta}^5\boldsymbol{\theta})$$

and

$$\omega_{7,0} = \frac{1}{35} \text{tr} \boldsymbol{\theta}^7.$$

Remark 2. The forms $\text{tr} \boldsymbol{\theta}^{2k-1}$, $k \geq 1$, where $\boldsymbol{\theta} = g^{-1}dg$ is a Maurer-Cartan form, generate the cohomology ring $H^1(\text{GL}(\infty, \mathbb{C}), \mathbb{Q})$ for the stabilized complex general linear group $\text{GL}(\infty, \mathbb{C})$.

3. DOUBLE ASCENT

3.1. Set-up. From descent equations it follows that there is a form $\omega_{2k-2,0}$ such that

$$\omega_{2k-1,0} = \partial\omega_{2k-2,0}.$$

Now going up from the bottom to the top (this explains the terminology), we get

$$\partial(\omega_{2k-2,1} + \bar{\partial}\omega_{2k-2,0}) = 0,$$

so that there is a form $\omega_{2k-3,1}$ such that

$$\omega_{2k-2,1} + \bar{\partial}\omega_{2k-2,0} = \partial\omega_{2k-3,1}.$$

Therefore

$$\partial(\omega_{2k-3,2} + \bar{\partial}\omega_{2k-3,1}) = 0$$

and there is a form $\omega_{2k-4,2}$ such that

$$\omega_{2k-3,2} + \bar{\partial}\omega_{2k-3,1} = \partial\omega_{2k-4,2}.$$

Repeating this procedure, we finally get a form $\omega_{k-1,k-1}$ such that

$$\omega_{k,k-1} + \bar{\partial}\omega_{k,k-2} = \partial\omega_{k-1,k-1}.$$

The ascent equations can be written succinctly as

$$k! \text{cs}_k = \partial\omega_{k-1,k-1} - d \left(\omega_{k,k-2} - \omega_{k+1,k-3} + \cdots + (-1)^k \omega_{2k-2,0} \right).$$

Defining CS_k as $\text{cs}_k \bmod \text{Im } d$ (see [11]), we get

$$(8) \quad \text{CS}_k = \frac{1}{k!} \partial\omega_{k-1,k-1}.$$

Therefore,

$$\text{ch}_k = \frac{1}{k!} \bar{\partial}\partial\omega_{k-1,k-1},$$

so that $\omega_{k-1,k-1}$ is a $k!$ times the Bott-Chern secondary form bc_k (see [3]).

Solving ‘explicitly’ ascent equations would give explicit local expression of the Chern character form ch_k in terms of the corresponding Bott-Chern form bc_k . It is known that it is not possible to get local formulas in terms

of the matrix h alone. This is because each step in the ascent procedure uses Poincaré lemma which, in general, contains an integration through the homotopy formula. However, one can solve these equations explicitly by using the Cholesky decomposition!

Namely, put

$$h = cb = b^*ab,$$

where matrix b is upper-triangular with 1's on the diagonal, and a is diagonal with positive entries; a_i and b_{ij} , $i = 1, \dots, r, j > i$, are global coordinates on the homogeneous space of hermitian positive-definite $r \times r$ matrices. We get

$$(9) \quad \theta = h^{-1}\partial h = b^{-1}\theta b = b^{-1}(\theta_1 + \theta_2)b,$$

where

$$\theta_1 = \partial b b^{-1} \quad \text{and} \quad \theta_2 = c^{-1}\partial c.$$

Therefore

$$(10) \quad \Theta = \bar{\partial}\theta = b^{-1}(\bar{\partial}\theta - \bar{\theta}_1\theta - \theta\bar{\theta}_1)b,$$

where

$$\bar{\theta}_1 = \bar{\partial} b b^{-1} \quad \text{and} \quad \bar{\theta}_2 = c^{-1}\bar{\partial} c.$$

These matrix-valued 1-forms satisfy

$$(11) \quad \partial\theta_1 = \theta_1^2, \quad \partial\theta_2 = -\theta_2^2, \quad \bar{\partial}\bar{\theta}_1 = \bar{\theta}_1^2, \quad \bar{\partial}\bar{\theta}_2 = -\bar{\theta}_2^2,$$

$$(12) \quad \bar{\partial}\theta_1 = -\partial\bar{\theta}_1 + \theta_1\bar{\theta}_1 + \bar{\theta}_1\theta_1, \quad \bar{\partial}\theta_2 = -\partial\bar{\theta}_2 - \theta_2\bar{\theta}_2 - \bar{\theta}_2\theta_2$$

and have important property

$$(13) \quad \text{tr } \theta_1^l = 0 \quad \text{for all } l \geq 1 \quad \text{and} \quad \text{tr } \theta_2^l = 0 \quad \text{for all } l \geq 2.$$

It turns out that in terms of $\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2$ and its ∂ and $\bar{\partial}$ differentials one can compute forms $\omega_{2k-2-l,l}$ for $l = 0, 1, \dots, k-1$.

We will be also using

$$(14) \quad \bar{\theta} = h^{-1}\bar{\partial}h = b^{-1}\bar{\theta}b = b^{-1}(\bar{\theta}_1 + \bar{\theta}_2)b,$$

so that

$$(15) \quad \bar{\partial}\bar{\theta} = -\bar{\theta}^2 \quad \text{and} \quad \Theta = -\partial\bar{\theta} - \theta\bar{\theta} - \bar{\theta}\theta.$$

Remark 3. The Cholesky decomposition is useful since by the holomorphic splitting principle (see, e.g., [12, Corollary 9.26]), for every holomorphic vector bundle $E \rightarrow X$ there exists a variety Y and a flat morphism $p : Y \rightarrow X$ such that the bundle $p^*(E)$ over Y admits upper-triangular transition functions.

3.2. The case $k = 2$. Start with the form $\omega_{3,0} = \frac{1}{3} \text{tr } \theta^3$. In terms of the Cholesky decomposition we have

$$\omega_{3,0} = \frac{1}{3} \text{tr}(\theta_1^3 + 3\theta_1^2\theta_2 + 3\theta_1\theta_2^2 + \theta_2^3) = \text{tr}(\theta_1^2\theta_2 + \theta_1\theta_2^2) = \partial \text{tr}(\theta_1\theta_2),$$

so that

$$\omega_{2,0} = \text{tr}(\theta_1\theta_2).$$

Using (15) we get

$$\begin{aligned} \omega_{2,1} &= \text{tr}(\theta\bar{\theta}) = -\text{tr}(\theta(\partial\bar{\theta} + \theta(\theta\bar{\theta} + \bar{\theta}\theta))) = \partial \text{tr}(\theta\bar{\theta}) + \text{tr}(\theta^2\bar{\theta} - \theta(\theta\bar{\theta} + \bar{\theta}\theta)) \\ &= \partial \text{tr}(\theta\bar{\theta}) - \text{tr}(\theta^2\bar{\theta}), \end{aligned}$$

and using (12) we obtain

$$\begin{aligned} \omega_{2,1} + \bar{\partial}\omega_{2,0} &= \partial \text{tr}(\theta\bar{\theta}) + \text{tr}(-\theta^2\bar{\theta} + (\bar{\partial}\theta_1\theta_2 - \theta_1\bar{\partial}\theta_2)) \\ &= \partial \text{tr}(\theta\bar{\theta}) + \text{tr}(-\theta^2\bar{\theta} - \partial\bar{\theta}_1\theta_2 + \theta_1\partial\bar{\theta}_2 + (\theta_1\theta_2 + \theta_2\theta_1)\bar{\theta}) \\ &= \partial \text{tr}(\theta\bar{\theta} - (\bar{\theta}_1\theta_2 - \bar{\theta}_2\theta_1)) \\ &\quad + \text{tr}(-\theta^2\bar{\theta} + \theta_1^2\bar{\theta}_2 + \bar{\theta}_2\bar{\theta}_1^2 + (\theta_1\theta_2 + \theta_2\theta_1)\bar{\theta}) \\ &= \partial \text{tr}(\theta\bar{\theta} - (\bar{\theta}_1\theta_2 - \bar{\theta}_2\theta_1)). \end{aligned}$$

Thus

$$\omega_{1,1} = \text{tr}(\theta\bar{\theta} - (\bar{\theta}_1\theta_2 - \bar{\theta}_2\theta_1)) = \text{tr}(2\theta_2\bar{\theta}_1 + \theta_2\bar{\theta}_2),$$

and we obtain the following result.

Proposition 2. *The second Bott-Chern form bc_2 of a trivial Hermitian vector bundle (\mathbb{C}^r, h) over a complex manifold X in Cholesky coordinates $h = b^*ab$ is given by the formula*

$$\text{bc}_2 = \frac{1}{2} \text{tr}(2\theta_2\bar{\theta}_1 + \theta_2\bar{\theta}_2).$$

Here $\bar{\theta}_1 = \bar{\partial}bb^{-1}$, $\theta_2 = c^{-1}\partial c$ and $\bar{\theta}_2 = c^{-1}\bar{\partial}c$.

Remark 4. We have

$$\omega_{1,1} = \text{tr}(2a^{-1}\partial a \wedge a^{-1}\bar{\partial}a + (\alpha\bar{\partial}bb^{-1}\alpha^{-1})^* \wedge (\alpha\bar{\partial}bb^{-1}\alpha^{-1})),$$

where $\alpha = \sqrt{a}$, from which it follows that $\sqrt{-1}\omega_{1,1} \geq 0$.

Remark 5. When

$$a = \begin{pmatrix} 1 & 0 \\ 0 & e^\sigma \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & \bar{f} \\ 0 & 1 \end{pmatrix},$$

we get

$$\omega_{0,0} = \sigma \quad \text{and} \quad \omega_{1,1} = \text{tr}(\partial\sigma \wedge \bar{\partial}\sigma + 2e^{-\sigma}\partial f \wedge \bar{\partial}\bar{f}),$$

so that

$$\frac{1}{2}\bar{\partial}\partial\omega_{1,1} - \frac{1}{2}(\bar{\partial}\partial\omega_{0,0})^2 = \bar{\partial}\partial(e^{-\sigma}\partial f \wedge \bar{\partial}\bar{f}),$$

in agreement with Remark 3.4 in [9].

3.3. **The case $k = 3$.** Using (13) we get

$$\begin{aligned}\omega_{5,0} &= \frac{1}{10} \operatorname{tr} \theta^5 = \frac{1}{10} \operatorname{tr} \theta^5 \\ &= \frac{1}{2} \operatorname{tr} (\theta_1^4 \theta_2 + \theta_1^3 \theta_2^2 + \theta_1^2 \theta_2 \theta_1 \theta_2 + \theta_1 \theta_2 \theta_1 \theta_2^2 + \theta_1^2 \theta_2^3 + \theta_1 \theta_2^4) \\ &= \frac{1}{2} \partial \operatorname{tr} \left(\theta_1^3 \theta_2 + \theta_1 \theta_2^3 + \frac{1}{2} (\theta_1 \theta_2)^2 \right),\end{aligned}$$

so that

$$\omega_{4,0} = \frac{1}{2} \operatorname{tr} \left(\theta_1^3 \theta_2 + \theta_1 \theta_2^3 + \frac{1}{2} (\theta_1 \theta_2)^2 \right).$$

We will compute $\omega_{4,1} + \bar{\partial} \omega_{4,0}$ and will find $\omega_{3,1}$ such that

$$\omega_{4,1} + \bar{\partial} \omega_{4,0} = \partial \omega_{3,1}.$$

First using (15) we get

$$\begin{aligned}\omega_{4,1} &= \frac{1}{2} \operatorname{tr} (\theta^3 \Theta) = -\frac{1}{2} \operatorname{tr} (\theta^3 (\partial \bar{\theta} + \bar{\theta} \theta + \theta \bar{\theta})) \\ &= \frac{1}{2} \partial \operatorname{tr} (\theta^3 \bar{\theta}) + \frac{1}{2} \operatorname{tr} (\theta^4 \bar{\theta} - \theta^3 (\bar{\theta} \theta + \theta \bar{\theta})) \\ &= \frac{1}{2} \partial \operatorname{tr} (\theta^3 \bar{\theta}) - \frac{1}{2} \operatorname{tr} (\theta^4 \bar{\theta}).\end{aligned}$$

Next, using (12) we obtain

$$\begin{aligned}\bar{\partial} \omega_{4,0} &= \frac{1}{2} \operatorname{tr} (I_1 \bar{\partial} \theta_1 + I_2 \bar{\partial} \theta_2) \\ &= \frac{1}{2} \operatorname{tr} (I_1 (-\partial \bar{\theta}_1 + \theta_1 \bar{\theta}_1 + \bar{\theta}_1 \theta_1) + I_2 (-\partial \bar{\theta}_2 - \theta_2 \bar{\theta}_2 - \bar{\theta}_2 \theta_2)) \\ &= \frac{1}{2} \partial \operatorname{tr} (I_1 \bar{\theta}_1 + I_2 \bar{\theta}_2) + \frac{1}{2} \operatorname{tr} ((-\partial I_1 + I_1 \theta_1 + \theta_1 I_1) \bar{\theta}_1 - (\partial I_2 + I_2 \theta_2 + \theta_2 I_2) \bar{\theta}_2),\end{aligned}$$

where

$$I_1 = \theta_1^3 + \theta_1^2 \theta_2 + \theta_2 \theta_1^2 - \theta_1 \theta_2 \theta_1 + \theta_2 \theta_1 \theta_2 + \theta_2^3 = \theta^3 - \theta \theta_2 \theta_1 - \theta_1 \theta_2 \theta$$

and

$$I_2 = -(\theta_1^3 + \theta_1 \theta_2 \theta_1 + \theta_1 \theta_2^2 - \theta_2 \theta_1 \theta_2 + \theta_2^2 \theta_1 + \theta_2^3) = -\theta^3 + \theta \theta_1 \theta_2 + \theta_2 \theta_1 \theta.$$

Using identities

$$\partial I_1 - I_1 \theta_1 - \theta_1 I_1 = -\theta^4 \quad \text{and} \quad \partial I_2 + I_2 \theta_2 + \theta_2 I_2 = -\theta^4,$$

we get

$$\omega_{4,1} + \bar{\partial} \omega_{4,0} = \frac{1}{2} \partial \operatorname{tr} (\theta^3 \bar{\theta} + I_1 \bar{\theta}_1 + I_2 \bar{\theta}_2),$$

so that

$$\omega_{3,1} = \frac{1}{2} \operatorname{tr} (\theta^3 \bar{\theta} + I_1 \bar{\theta}_1 + I_2 \bar{\theta}_2).$$

Equivalently,

$$\omega_{3,1} = \frac{1}{2} \operatorname{tr}(2\theta^3\bar{\theta}_1 - (\theta_1\theta_2^2 + 2\theta_1\theta_2\theta_1 + \theta_2^2\theta_1)\bar{\theta}_1 + (\theta_1^2\theta_2 + 2\theta_2\theta_1\theta_2 + \theta_2\theta_1^2)\bar{\theta}_2).$$

Finally, we will compute $\omega_{3,2} + \bar{\partial}\omega_{3,1}$ and find $\omega_{2,2}$ such that

$$\omega_{3,2} + \bar{\partial}\omega_{3,1} = \partial\omega_{2,2}.$$

First, using (15) we obtain

$$\begin{aligned} \partial \operatorname{tr}(\theta\Theta\bar{\theta}) &= \operatorname{tr}(-\theta^2\Theta\bar{\theta} - \theta(\Theta\theta - \theta\Theta)\bar{\theta} + \theta\Theta(\Theta + \theta\bar{\theta} + \bar{\theta}\theta)) \\ &= \operatorname{tr}(\theta\Theta^2 + \theta^2\Theta\bar{\theta}), \end{aligned}$$

and

$$\begin{aligned} \partial \operatorname{tr}(\bar{\theta}\Theta\theta) &= \operatorname{tr}(-(\Theta + \theta\bar{\theta} + \bar{\theta}\theta)\Theta\theta - \bar{\theta}(\Theta\theta - \theta\Theta)\theta + \bar{\theta}\Theta\theta^2) \\ &= -\operatorname{tr}(\theta\Theta^2 + \bar{\theta}\Theta\theta^2), \end{aligned}$$

so that

$$\omega_{3,2} = \operatorname{tr}(\theta\Theta^2) = \partial \left\{ \frac{1}{2} \operatorname{tr}(\theta\Theta\bar{\theta} - \bar{\theta}\Theta\theta) \right\} - \frac{1}{2} \operatorname{tr}(\theta^2\Theta\bar{\theta} + \bar{\theta}\Theta\theta^2).$$

Next, we write

$$\omega_{3,1} = \frac{1}{2} \operatorname{tr}(\theta^3\bar{\theta} + I_1\bar{\theta}_1 + I_2\bar{\theta}_2) = \omega_{3,1}^{(1)} + \omega_{3,1}^{(2)},$$

where $\bar{\theta}_1 = b^{-1}\bar{\theta}_1b$ and $\bar{\theta}_2 = b^{-1}\bar{\theta}_2b$ and

$$I_1 = \theta^3 - \theta\theta_2\theta_1 - \theta_1\theta_2\theta, \quad I_2 = -\theta^3 + \theta\theta_1\theta_2 + \theta_2\theta_1\theta,$$

where $\theta_1 = b^{-1}\theta_1b$ and $\theta_2 = b^{-1}\theta_2b$. We have

$$\begin{aligned} \bar{\partial}\omega_{3,1}^{(1)} &= \frac{1}{2} \bar{\partial} \operatorname{tr}(\theta^3\bar{\theta}) \\ &= \frac{1}{2} \operatorname{tr}(\Theta\theta^2\bar{\theta} - \theta\Theta\theta\bar{\theta} + \theta^2\Theta\bar{\theta} + \theta^3\bar{\theta}^2) \\ &= \frac{1}{2} \operatorname{tr}(\bar{\theta}\Theta\theta^2 + \theta^2\Theta\bar{\theta} - \theta\Theta\theta\bar{\theta} + \theta^3\bar{\theta}^2), \end{aligned}$$

so that

$$\omega_{3,2} + \bar{\partial}\omega_{3,1}^{(1)} = \partial \left\{ \frac{1}{2} \operatorname{tr}(\theta\Theta\bar{\theta} - \bar{\theta}\Theta\theta) \right\} + \frac{1}{2} \operatorname{tr}(\theta^3\bar{\theta}^2 - \theta\bar{\theta}\theta\Theta).$$

We also have

$$\begin{aligned} \partial \operatorname{tr}(\theta\bar{\theta})^2 &= 2 \operatorname{tr}((- \theta^2\bar{\theta} - \theta\partial\bar{\theta})\theta\bar{\theta}) = 2 \operatorname{tr}((- \theta^2\bar{\theta} + \theta\Theta + \theta\theta\bar{\theta} + \theta\bar{\theta}\theta)\theta\bar{\theta}) \\ &= 2 \operatorname{tr}(\theta\Theta\theta\bar{\theta} + \theta\bar{\theta}\theta\theta\bar{\theta}), \end{aligned}$$

so that

$$\operatorname{tr}(\theta\bar{\theta}\theta\Theta) = \partial \left\{ \frac{1}{2} \operatorname{tr}(\theta\bar{\theta})^2 \right\} - \operatorname{tr}(\theta^2\bar{\theta}\theta\bar{\theta}).$$

Thus we obtain

$$\omega_{3,2} + \bar{\partial}\omega_{3,1}^{(1)} = \partial \left\{ \frac{1}{2} \operatorname{tr} \left(\boldsymbol{\theta} \boldsymbol{\Theta} \bar{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}} \boldsymbol{\Theta} \boldsymbol{\theta} - \frac{1}{2} (\boldsymbol{\theta} \bar{\boldsymbol{\theta}})^2 \right) \right\} + \frac{1}{2} \operatorname{tr} (\boldsymbol{\theta}^3 \bar{\boldsymbol{\theta}}^2 + \boldsymbol{\theta}^2 \bar{\boldsymbol{\theta}} \boldsymbol{\theta} \bar{\boldsymbol{\theta}}).$$

So far we have not used the Cholesky decomposition and now we start using it for this case. Note that

$$\omega_{3,2} + \bar{\partial}\omega_{3,1}^{(1)} = \partial \left\{ \frac{1}{2} \operatorname{tr} \left(\boldsymbol{\theta} \boldsymbol{\Theta} \bar{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}} \boldsymbol{\Theta} \boldsymbol{\theta} - \frac{1}{2} (\boldsymbol{\theta} \bar{\boldsymbol{\theta}})^2 \right) \right\} + \frac{1}{2} \operatorname{tr} (\boldsymbol{\theta}^3 \bar{\boldsymbol{\theta}}^2 + \boldsymbol{\theta}^2 \bar{\boldsymbol{\theta}} \boldsymbol{\theta} \bar{\boldsymbol{\theta}})$$

and it remains to compute

$$\begin{aligned} \bar{\partial}\omega_{3,1}^{(2)} &= \frac{1}{2} \bar{\partial} \operatorname{tr} (\boldsymbol{I}_1 \bar{\boldsymbol{\theta}}_1 + \boldsymbol{I}_2 \bar{\boldsymbol{\theta}}_2) = \frac{1}{2} \bar{\partial} \operatorname{tr} (I_1 \bar{\theta}_1 + I_2 \bar{\theta}_2) \\ &= \frac{1}{2} \operatorname{tr} (\bar{\partial} I_1 \bar{\theta}_1 + \bar{\partial} I_2 \bar{\theta}_2 - I_1 \bar{\theta}_1^2 + I_2 \bar{\theta}_2^2). \end{aligned}$$

By a straightforward computation using

$$\bar{\partial}\boldsymbol{\theta} = -\partial\bar{\boldsymbol{\theta}} + \boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 - \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2$$

we get

$$\begin{aligned} \bar{\partial} I_1 \bar{\theta}_1 &= \\ &= \operatorname{tr} \{ [\bar{\partial}\boldsymbol{\theta}\boldsymbol{\theta}^2 - \boldsymbol{\theta}\bar{\partial}\boldsymbol{\theta}\boldsymbol{\theta} + \boldsymbol{\theta}^2 \bar{\partial}\boldsymbol{\theta} - \bar{\partial}\boldsymbol{\theta}\boldsymbol{\theta}_2 \boldsymbol{\theta}_1 + \boldsymbol{\theta}(\bar{\partial}\boldsymbol{\theta}_2 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\partial}\boldsymbol{\theta}_1) - \\ &\quad - (\bar{\partial}\boldsymbol{\theta}_1 \boldsymbol{\theta}_2 - \boldsymbol{\theta}_1 \bar{\partial}\boldsymbol{\theta}_2) \boldsymbol{\theta} - \boldsymbol{\theta}_1 \boldsymbol{\theta}_2 \bar{\partial}\boldsymbol{\theta}] \bar{\boldsymbol{\theta}}_1 \} \\ &= \operatorname{tr} \{ [-\bar{\partial}\bar{\boldsymbol{\theta}}\boldsymbol{\theta}^2 + \boldsymbol{\theta}\bar{\partial}\bar{\boldsymbol{\theta}}\boldsymbol{\theta} - \boldsymbol{\theta}^2 \bar{\partial}\bar{\boldsymbol{\theta}} + \bar{\partial}\bar{\boldsymbol{\theta}}\boldsymbol{\theta}_2 \boldsymbol{\theta}_1 + \boldsymbol{\theta}(\boldsymbol{\theta}_2 \bar{\partial}\bar{\boldsymbol{\theta}}_1 - \bar{\partial}\bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_1) + \\ &\quad + (\bar{\partial}\bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_2 - \boldsymbol{\theta}_1 \bar{\partial}\bar{\boldsymbol{\theta}}_2) \boldsymbol{\theta} + \boldsymbol{\theta}_1 \boldsymbol{\theta}_2 \bar{\partial}\bar{\boldsymbol{\theta}}] \bar{\boldsymbol{\theta}}_1 \} + \\ &+ \operatorname{tr} \{ [(\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 - \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2) \boldsymbol{\theta}^2 - \boldsymbol{\theta}(\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 - \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2) \boldsymbol{\theta} + \\ &\quad + \boldsymbol{\theta}^2 (\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 - \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2) - (\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 - \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2) \boldsymbol{\theta}_2 \boldsymbol{\theta}_1 - \\ &\quad - \boldsymbol{\theta}((\boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 + \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2) \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 (\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1)) - ((\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1) \boldsymbol{\theta}_2 + \boldsymbol{\theta}_1 (\boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 + \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2)) \boldsymbol{\theta} - \\ &\quad - \boldsymbol{\theta}_1 \boldsymbol{\theta}_2 (\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 - \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2)] \bar{\boldsymbol{\theta}}_1 \} \end{aligned}$$

and

$$\begin{aligned} \bar{\partial} I_2 \bar{\theta}_2 &= \\ &= \operatorname{tr} \{ [-\bar{\partial}\bar{\boldsymbol{\theta}}\boldsymbol{\theta}^2 + \boldsymbol{\theta}\bar{\partial}\bar{\boldsymbol{\theta}}\boldsymbol{\theta} - \boldsymbol{\theta}^2 \bar{\partial}\bar{\boldsymbol{\theta}} + \bar{\partial}\bar{\boldsymbol{\theta}}\boldsymbol{\theta}_1 \boldsymbol{\theta}_2 - \boldsymbol{\theta}(\bar{\partial}\boldsymbol{\theta}_1 \boldsymbol{\theta}_2 - \boldsymbol{\theta}_1 \bar{\partial}\boldsymbol{\theta}_2) + \\ &\quad + (\bar{\partial}\boldsymbol{\theta}_2 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\partial}\boldsymbol{\theta}_1) \boldsymbol{\theta} + \boldsymbol{\theta}_2 \boldsymbol{\theta}_1 \bar{\partial}\bar{\boldsymbol{\theta}}] \bar{\boldsymbol{\theta}}_2 \} \\ &= \operatorname{tr} \{ [\bar{\partial}\bar{\boldsymbol{\theta}}\boldsymbol{\theta}^2 - \boldsymbol{\theta}\bar{\partial}\bar{\boldsymbol{\theta}}\boldsymbol{\theta} + \boldsymbol{\theta}^2 \bar{\partial}\bar{\boldsymbol{\theta}} - \bar{\partial}\bar{\boldsymbol{\theta}}\boldsymbol{\theta}_1 \boldsymbol{\theta}_2 - \boldsymbol{\theta}(\boldsymbol{\theta}_1 \bar{\partial}\bar{\boldsymbol{\theta}}_2 - \bar{\partial}\bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_2) - \\ &\quad - (\bar{\partial}\bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\partial}\bar{\boldsymbol{\theta}}_1) \boldsymbol{\theta} - \boldsymbol{\theta}_2 \boldsymbol{\theta}_1 \bar{\partial}\bar{\boldsymbol{\theta}}] \bar{\boldsymbol{\theta}}_2 \} + \\ &+ \operatorname{tr} \{ [-(\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 - \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2) \boldsymbol{\theta}^2 + \boldsymbol{\theta}(\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 - \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2) \boldsymbol{\theta} - \\ &\quad - \boldsymbol{\theta}^2 (\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 - \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2) + (\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 - \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2) \boldsymbol{\theta}_1 \boldsymbol{\theta}_2 - \\ &\quad - \boldsymbol{\theta}(\boldsymbol{\theta}_1 (\boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 + \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2) + (\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1) \boldsymbol{\theta}_2) - (\boldsymbol{\theta}_2 (\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1) + (\boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 + \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2) \boldsymbol{\theta}_1) \boldsymbol{\theta} + \\ &\quad + \boldsymbol{\theta}_2 \boldsymbol{\theta}_1 (\boldsymbol{\theta}_1 \bar{\boldsymbol{\theta}}_1 + \bar{\boldsymbol{\theta}}_1 \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \bar{\boldsymbol{\theta}}_2 - \bar{\boldsymbol{\theta}}_2 \boldsymbol{\theta}_2)] \bar{\boldsymbol{\theta}}_2 \}. \end{aligned}$$

Thus we obtain

$$\text{tr}(\bar{\partial}I_1\bar{\theta}_1 + \bar{\partial}I_2\bar{\theta}_2) = J_1 + J_2,$$

where

$$\begin{aligned} J_1 = & \text{tr}\{ -\bar{\partial}\bar{\theta}_2(\theta_1^2 + \theta_2^2 + \theta_1\theta_2)\bar{\theta}_1 + (\theta_1^2 + \theta_2^2 + \theta_1\theta_2)\partial\theta_1\bar{\theta}_2 - (\theta_1^2 + \theta_2^2 + \theta_2\theta_1)\partial\bar{\theta}_2\bar{\theta}_1 \\ & + \partial\bar{\theta}_1(\theta_1^2 + \theta_2^2 + \theta_2\theta_1)\bar{\theta}_2 + \partial\bar{\theta}_1(\theta_2\theta_1 - \theta_1\theta_2)\bar{\theta}_1 - (\theta_2\theta_1 - \theta_1\theta_2)\partial\bar{\theta}_1\bar{\theta}_1 \\ & + \partial\bar{\theta}_2(\theta_2\theta_1 - \theta_1\theta_2)\bar{\theta}_2 - (\theta_2\theta_1 - \theta_1\theta_2)\partial\bar{\theta}_2\bar{\theta}_2 + \theta_2\partial\bar{\theta}_1\theta_2\bar{\theta}_1 + \theta_2\partial\bar{\theta}_2\theta_2\bar{\theta}_1 \\ & - \theta_1\partial\bar{\theta}_1\theta_1\bar{\theta}_2 - \theta_1\partial\bar{\theta}_2\theta_1\bar{\theta}_2 + \theta_1\partial\bar{\theta}_1\theta_2\bar{\theta}_1 + \theta_2\partial\bar{\theta}_1\theta_1\bar{\theta}_1 \\ & - \theta_1\partial\bar{\theta}_2\theta_2\bar{\theta}_2 - \theta_2\partial\bar{\theta}_2\theta_1\bar{\theta}_2 - \theta_1\partial\bar{\theta}_2\theta_1\bar{\theta}_1 + \theta_2\partial\bar{\theta}_1\theta_2\bar{\theta}_2 \} \end{aligned}$$

and

$$\begin{aligned} J_2 = & \text{tr}\{ [(\theta_1\bar{\theta}_1 + \bar{\theta}_1\theta_1 - \theta_2\bar{\theta}_2 - \bar{\theta}_2\theta_2)\theta^2 - \theta(\theta_1\bar{\theta}_1 + \bar{\theta}_1\theta_1 - \theta_2\bar{\theta}_2 - \bar{\theta}_2\theta_2)\theta \\ & + \theta^2(\theta_1\bar{\theta}_1 + \bar{\theta}_1\theta_1 - \theta_2\bar{\theta}_2 - \bar{\theta}_2\theta_2)] (\bar{\theta}_1 - \bar{\theta}_2) - (\theta_1\bar{\theta}_1 + \bar{\theta}_1\theta_1 - \theta_2\bar{\theta}_2 - \bar{\theta}_2\theta_2)(\theta_2\theta_1\bar{\theta}_1 - \theta_1\theta_2\bar{\theta}_2) \\ & - \theta_1\theta_2(\theta_1\bar{\theta}_1 + \bar{\theta}_1\theta_1 - \theta_2\bar{\theta}_2 - \bar{\theta}_2\theta_2)\bar{\theta}_1 + \theta_2\theta_1(\theta_1\bar{\theta}_1 + \bar{\theta}_1\theta_1 - \theta_2\bar{\theta}_2 - \bar{\theta}_2\theta_2)\bar{\theta}_2 \\ & - \theta((\theta_2\bar{\theta}_2 + \bar{\theta}_2\theta_2)\theta_1 + \theta_2(\theta_1\bar{\theta}_1 + \bar{\theta}_1\theta_1))\bar{\theta}_1 - \theta(\theta_1(\theta_2\bar{\theta}_2 + \bar{\theta}_2\theta_2) + (\theta_1\bar{\theta}_1 + \bar{\theta}_1\theta_1)\theta_2)\bar{\theta}_2 \\ & - (\theta_1(\theta_2\bar{\theta}_2 + \bar{\theta}_2\theta_2) + (\theta_1\bar{\theta}_1 + \bar{\theta}_1\theta_1)\theta_2)\theta\bar{\theta}_1 - ((\theta_2\bar{\theta}_2 + \bar{\theta}_2\theta_2)\theta_1 + \theta_2(\theta_1\bar{\theta}_1 + \bar{\theta}_1\theta_1)\theta)\bar{\theta}_2 \}. \end{aligned}$$

Simplifying and using the cyclic property of the trace, we get

$$\begin{aligned} J_1 = & \text{tr}\{ (\theta_1^2 + \theta_2^2 + \theta_1\theta_2)\partial(\bar{\theta}_1\bar{\theta}_2) - (\theta_1^2 + \theta_2^2 + \theta_2\theta_1)\partial(\bar{\theta}_2\bar{\theta}_1) - (\theta_2\theta_1 - \theta_1\theta_2)\partial(\bar{\theta}_1^2 + \bar{\theta}_2^2) \\ & + (\theta_2\partial\bar{\theta}_1\theta_2\bar{\theta}_2 + \theta_2\bar{\theta}_1\theta_2\partial\bar{\theta}_2) - (\theta_1\partial\bar{\theta}_1\theta_1\bar{\theta}_2 + \theta_1\bar{\theta}_1\theta_1\partial\bar{\theta}_2) + (\theta_2\partial\bar{\theta}_1\theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_1\theta_1\partial\bar{\theta}_1) \\ & - (\theta_2\partial\bar{\theta}_2\theta_1\bar{\theta}_2 + \theta_2\bar{\theta}_2\theta_1\partial\bar{\theta}_2) + \theta_2\partial\bar{\theta}_1\theta_2\bar{\theta}_1 - \theta_1\partial\bar{\theta}_2\theta_1\bar{\theta}_2 \} \\ = & \partial\{ \text{tr}((\theta_1^2 + \theta_2^2 + \theta_1\theta_2)\bar{\theta}_1\bar{\theta}_2 - (\theta_1^2 + \theta_2^2 + \theta_2\theta_1)\bar{\theta}_2\bar{\theta}_1 - (\theta_2\theta_1 - \theta_1\theta_2)(\bar{\theta}_1^2 + \bar{\theta}_2^2) \\ & - \theta_2\bar{\theta}_1\theta_2\bar{\theta}_2 + \theta_1\bar{\theta}_1\theta_1\bar{\theta}_2 - \theta_2\bar{\theta}_1\theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2\theta_1\bar{\theta}_2 + \frac{1}{2}((\theta_1\bar{\theta}_2)^2 - (\theta_2\bar{\theta}_1)^2)) \} \\ & + \text{tr}\{ -(\theta_1^2\theta_2 + \theta_1\theta_2^2)\bar{\theta}_1\bar{\theta}_2 - (\theta_2^2\theta_1 + \theta_2\theta_1^2)\bar{\theta}_2\bar{\theta}_1 - (\theta_1^2\theta_2 + \theta_1\theta_2^2 + \theta_2^2\theta_1 + \theta_2\theta_1^2)(\bar{\theta}_1^2 + \bar{\theta}_2^2) \\ & - \theta_2^2\bar{\theta}_1\theta_2\bar{\theta}_2 - \theta_2^2\bar{\theta}_2\theta_2\bar{\theta}_1 - \theta_1^2\bar{\theta}_1\theta_1\bar{\theta}_2 - \theta_1^2\bar{\theta}_2\theta_1\bar{\theta}_1 - \theta_2^2\bar{\theta}_1\theta_1\bar{\theta}_1 + \theta_1^2\bar{\theta}_1\theta_2\bar{\theta}_1 \\ & + \theta_2^2\bar{\theta}_2\theta_1\bar{\theta}_2 - \theta_1^2\bar{\theta}_2\theta_2\bar{\theta}_2 - \theta_1^2\bar{\theta}_2\theta_1\bar{\theta}_2 - \theta_2^2\bar{\theta}_1\theta_2\bar{\theta}_1 \} = J_{11} + J_{12}, \end{aligned}$$

where

$$\begin{aligned} J_{11} = & \partial\{ \text{tr}((\theta_1^2 + \theta_2^2 + \theta_1\theta_2)\bar{\theta}_1\bar{\theta}_2 - (\theta_1^2 + \theta_2^2 + \theta_2\theta_1)\bar{\theta}_2\bar{\theta}_1 - (\theta_2\theta_1 - \theta_1\theta_2)(\bar{\theta}_1^2 + \bar{\theta}_2^2) \\ & - \theta_2\bar{\theta}_1\theta_2\bar{\theta}_2 + \theta_1\bar{\theta}_1\theta_1\bar{\theta}_2 - \theta_2\bar{\theta}_1\theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2\theta_1\bar{\theta}_2 + \frac{1}{2}((\theta_1\bar{\theta}_2)^2 - (\theta_2\bar{\theta}_1)^2)) \} \end{aligned}$$

and

$$\begin{aligned} J_{12} = & \text{tr}\{ -(\theta_1^2\theta_2 + \theta_1\theta_2^2)\bar{\theta}_1\bar{\theta}_2 - (\theta_2^2\theta_1 + \theta_2\theta_1^2)\bar{\theta}_2\bar{\theta}_1 - (\theta_1^2\theta_2 + \theta_1\theta_2^2 + \theta_2^2\theta_1 + \theta_2\theta_1^2)(\bar{\theta}_1^2 + \bar{\theta}_2^2) \\ & - \theta_2^2\bar{\theta}_1\theta_2\bar{\theta}_2 - \theta_2^2\bar{\theta}_2\theta_2\bar{\theta}_1 - \theta_1^2\bar{\theta}_1\theta_1\bar{\theta}_2 - \theta_1^2\bar{\theta}_2\theta_1\bar{\theta}_1 - \theta_2^2\bar{\theta}_1\theta_1\bar{\theta}_1 + \theta_1^2\bar{\theta}_1\theta_2\bar{\theta}_1 \\ & + \theta_2^2\bar{\theta}_2\theta_1\bar{\theta}_2 - \theta_1^2\bar{\theta}_2\theta_2\bar{\theta}_2 - \theta_1^2\bar{\theta}_2\theta_1\bar{\theta}_2 - \theta_2^2\bar{\theta}_1\theta_2\bar{\theta}_1 \}, \end{aligned}$$

and

$$\begin{aligned}
J_2 = \text{tr} \{ & (\theta^2(\bar{\theta}_1 - \bar{\theta}_2)(\theta_1\bar{\theta}_1 - \theta_2\bar{\theta}_2) + \theta_1\theta^2(\bar{\theta}_1 - \bar{\theta}_2)\bar{\theta}_1 - \theta_2\theta^2(\bar{\theta}_1 - \bar{\theta}_2)\bar{\theta}_2 \\
& - \theta(\theta_1\bar{\theta}_1 - \theta_2\bar{\theta}_2)\theta(\bar{\theta}_1 - \bar{\theta}_2) - \theta_1\theta(\bar{\theta}_1 - \bar{\theta}_2)\theta\bar{\theta}_1 + \theta_2\theta(\bar{\theta}_1 - \bar{\theta}_2)\theta\bar{\theta}_2 \\
& + \theta^2(\theta_1\bar{\theta}_1 - \theta_2\bar{\theta}_2)(\bar{\theta}_1 - \bar{\theta}_2) + \theta^2(\bar{\theta}_1\theta_1 - \bar{\theta}_2\theta_2)(\bar{\theta}_1 - \bar{\theta}_2) - \theta_2\theta_1\bar{\theta}_1\theta_1\bar{\theta}_1 \\
& - \theta_1\theta_2\theta_1\bar{\theta}_1^2 + \theta_2\theta_1\bar{\theta}_1\theta_2\bar{\theta}_2 + \theta_2\theta_2\theta_1\bar{\theta}_1\bar{\theta}_2 + \theta_1\theta_2\bar{\theta}_2\theta_1\bar{\theta}_1 + \theta_1^2\theta_2\bar{\theta}_2\bar{\theta}_1 \\
& - \theta_1\theta_2\bar{\theta}_2\theta_2\bar{\theta}_2 - \theta_2\theta_1\theta_2\bar{\theta}_2^2 - \theta_1\theta_2(\theta_1\bar{\theta}_1 - \theta_2\bar{\theta}_2)\bar{\theta}_1 - \theta_1\theta_2(\bar{\theta}_1\theta_1 - \bar{\theta}_2\theta_2)\bar{\theta}_1 \\
& + \theta_2\theta_1(\theta_1\bar{\theta}_1 - \theta_2\bar{\theta}_2)\bar{\theta}_2 + \theta_2\theta_1(\bar{\theta}_1\theta_1 - \bar{\theta}_2\theta_2)\bar{\theta}_2 - \theta\theta_2\bar{\theta}_1\bar{\theta}_1 - 2\theta_2\theta_1\bar{\theta}_1\theta\bar{\theta}_2 \\
& - \theta\theta_2\bar{\theta}_1\bar{\theta}_1^2 - \theta\theta_1\bar{\theta}_2\bar{\theta}_2 - \theta\theta_1\theta_2\bar{\theta}_2^2 - 2\theta_1\theta_2\bar{\theta}_2\theta\bar{\theta}_1 - \theta_2\theta\bar{\theta}_1\theta_1\bar{\theta}_2 - \theta_2\theta\bar{\theta}_1\theta_1\bar{\theta}_1 \\
& - \theta_1\theta_2\theta\bar{\theta}_1^2 - \theta_1\theta\bar{\theta}_2\theta_2\bar{\theta}_2 - \theta_2\theta_1\theta\bar{\theta}_2^2 - \theta_1\theta\bar{\theta}_2\theta_2\bar{\theta}_1 \}.
\end{aligned}$$

Simplifying $J_{12} + J_2$ once again and after using numerous ‘miraculous cancellations’, we obtain

$$\tilde{J}_1 + J_2 + \text{tr}(-I_1\bar{\theta}_1^2 + I_2\bar{\theta}_2^2) = -\text{tr}(\theta^3\bar{\theta}^2 + \theta^2\bar{\theta}\theta\bar{\theta}),$$

so that finally

$$\begin{aligned}
\omega_{3,2} + \bar{\omega}_{3,1} = \partial \Big\{ & \frac{1}{2} \text{tr}(\theta\theta\bar{\theta} - \bar{\theta}\theta\theta - \frac{1}{2}(\theta\bar{\theta})^2 + (\theta_1^2 + \theta_2^2 + \theta_1\theta_2)\bar{\theta}_1\bar{\theta}_2 \\
& - (\theta_1^2 + \theta_2^2 + \theta_2\theta_1)\bar{\theta}_2\bar{\theta}_1 - (\theta_2\theta_1 - \theta_1\theta_2)(\bar{\theta}_1^2 + \bar{\theta}_2^2) - \theta_2\bar{\theta}_1\theta_2\bar{\theta}_2 \\
& + \theta_1\bar{\theta}_1\theta_1\bar{\theta}_2 - \theta_2\bar{\theta}_1\theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2\theta_1\bar{\theta}_2 + \frac{1}{2}((\theta_1\bar{\theta}_2)^2 - (\theta_2\bar{\theta}_1)^2) \Big\}.
\end{aligned}$$

Thus we obtain the following result.

Proposition 3. *The third Bott-Chern form bc_3 of a trivial Hermitian vector bundle (\mathbb{C}^r, h) over a complex manifold X in Cholesky coordinates $h = b^*ab$ is given by the formula*

$$\begin{aligned}
\text{bc}_3 = \frac{1}{12} \text{tr} \Big(& \theta\theta\bar{\theta} - \bar{\theta}\theta\theta - \frac{1}{2}(\theta\bar{\theta})^2 + (\theta_1^2 + \theta_2^2 + \theta_1\theta_2)\bar{\theta}_1\bar{\theta}_2 \\
& - (\theta_1^2 + \theta_2^2 + \theta_2\theta_1)\bar{\theta}_2\bar{\theta}_1 - (\theta_2\theta_1 - \theta_1\theta_2)(\bar{\theta}_1^2 + \bar{\theta}_2^2) - \theta_2\bar{\theta}_1\theta_2\bar{\theta}_2 \\
& + \theta_1\bar{\theta}_1\theta_1\bar{\theta}_2 - \theta_2\bar{\theta}_1\theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2\theta_1\bar{\theta}_2 + \frac{1}{2}((\theta_1\bar{\theta}_2)^2 - (\theta_2\bar{\theta}_1)^2) \Big).
\end{aligned}$$

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